SOLUTION OF A PROBLEM CONCERNING THE MOTION OF VISCOELASTIC MATERIALS

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We give a general solution of a problem involving plane-parallel shearing motion of a Maxwell body in a plastometer subject to a quasistatic concentrated interaction. In our solution we employ physicomathematical modelling of the microstructure and creep processes in the distributed mass of viscoelastic materials.

In studying the physicomechanical properties of viscoelastic materials and their characteristics, as, for example, the coefficient of viscosity, the shear modulus, and the relaxation time, use is often made of a plastometer for plane-parallel shear [1, 2]. A plastometer is a system of two rigid steel plates between which is included the material mass being studied (see Fig. 1). The shearing force is established by a constant load P by means of a steel wire and a guiding pully with moment of inertia I and outer radius R. In the modelling of the microstructure and the creep processes in viscoelastic materials no account is taken, in the majority of cases, of the inertia of the distributed mass of the specimen, the inertia of the apparent additional masses, and the local creep [2-6].

An attempt was made in [7] to take into account the inertia of the distributed mass in the equations of motion of a Maxwell body under a concentrated shearing force. The limit of the creep function (23) in [7], under the condition that the viscosity coefficient $\eta \to \infty$, does not agree with the solution of the analogous problem in [8] for an absolutely elastic system.

With regard to relaxational and oscillational processes inside the moving material (despite the fact that the load P is constant), we can state that the force on the moving plate of the plastometer can vary depending on the nature of these processes. These phenomena were not taken into account in the analysis of specific experimental data. In the present paper we solve a problem involving the motion of a Maxwell body in a plane-parallel shear plastometer subject to quasistatic interactions; we take into account the inertia of the distributed mass of the material under study and also the inertia of the apparent additional mass, the latter being an industrial characteristic of the instrumentation; also taken into account is the change in the external shearing force with motion of the system.

We consider a plane-parallel shear plastometer, as shown in the figure: 1 is the pulley support, and 2 is the pulley; M_0 is the mass of the upper plate and the attached rod; P is the constant shearing load; U(z, t) is the displacement function for an infinitely thin horizontal layer of the material, the displacement being in the direction of the shearing force; z is a coordinate axis; h is the distance between the plastometer plates (the material layer thickness); the lower plastometer plate is rigidly fastened (oblique shading).

An indicator rod is rigidly fastened to the moving plate. Its common mass M, in the absence of an external load, plays the role of an apparent additional mass. When the system is subjected to the load P the apparent additional mass $M = M_0 + P/g + I/R^2$, where P/g is the mass for the load, I/R^2 is the mass of the moving pulley, g is the gravitational acceleration, and M_0, P, I, and R are as defined above. We neglect the mass of the wire.

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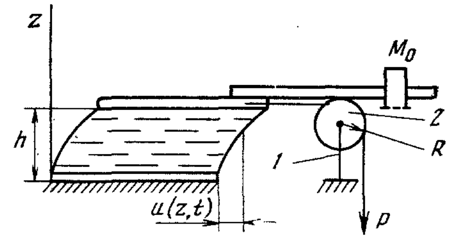

Fig. 1

Next we consider the behavior of the plastometer containing a viscoelastic material in its natural state [9]. We take a simple constitutive model, namely, the Maxwell model. It is characterized by two physical quantities: the shear modulus G and the viscosity coefficient η for the material under study. We select the experimental conditions such that G and η can be regarded as constants throughout the specimen. The local relationship between the relative shear γ and the mechanical shear stress σ can be written in the form

$$\eta\dot{\gamma} = \sigma + \tau\dot{\sigma} \tag{1}$$

where τ = η/G is the relaxation-time constant.

The rigidity of the specimen is much less than the rigidity of the steel plastometer components. We neglect deformations of the latter and assume that all the elements of the mass M are displaced in time with the same speed and acceleration and remain mutually parallel. We can regard the mass M as being concentrated in an infinitely thin layer at the upper boundary S(h) of the specimen, where S(h) is the upper contact area between the plate and the specimen. Consequently, the displacement, speed, and acceleration of the mass M will coincide with U(h, t), U_t(h, t), and U_{tt}(h, t), respectively, where U(h, t) is the displacement of the upper plate, i.e., the contact layer between the plate and the specimen.

Under the action of the load P on the specimen surface S(h) we have, as η → ∞, the stress

$$\sigma(h, t) = \frac{P}{S} - \frac{U_{tt}(h, t)}{S}\left[\frac{P}{g} + \frac{I}{R^2} + M_0\right] - \frac{f}{S} \tag{2}$$

where f is the force arising from external friction in the pulley bearings and internal friction of the wire. In the plastometers referred to, the force f almost never exceeds 2Γ, which amounts to a negligibly small quantity in relation to the applied load P, so that we can neglect it. We take the area of a cross section of the specimen by a horizontal plane in the direction of the shearing force to be constant throughout the region 0 ≤ z ≤ h. The creep process is assumed to be isothermal owing to the fact that the specimen thickness is 1 to 2 mm, and the mean rate of shear is small. The specimen length and the external shear force are selected so that the motion of the specimen can be regarded as a simple shear [7].

Under these assumptions the motion of the system may be described by a function U(z, t), giving the horizontal displacement of an infinitely thin layer of the material at height z, the relative shear of the layer being given by

$$\gamma = \partial U(z, t) / \partial z \tag{3}$$

Using the relations (1)-(3) and the physico-mathematical modelling of [9, 10], we write the partial differential equation for the displacement function:

$$U_{ttt} + \frac{1}{\tau} U_{tt} - \frac{G}{\rho} U_{tzz} = 0 \tag{4}$$

where ρ is the density of the material.

For t < 0 we assume there is no load (P = 0); for t > 0, we assume P = const, so that

$$U(z, 0) = 0, \quad U_t(z, 0) = 0, \quad 0 \leqslant z \leqslant h \tag{5}$$

Since the frictional force in the system is proportional to the displacement speed U_t(z, t) and U_t(z, 0) = 0 we can take as a third initial condition the relations

$$\rho U_{tt}(z, 0) - G\, U_{zz}(z, 0) = 0 \quad \text{for } z < h \tag{6}$$

and for z = h

$$U_z(h, 0) = \frac{P}{SG} - \frac{M}{SG} U_{tt}(h, 0) \tag{7}$$

The boundary conditions of the moving system can be determined for $t > 0$ as in [8, 11]. starting from the equilibrium of the forces close to the boundary as $z \to h$ and the relations (1)-(3). We write them in the form

$$U(0, t) = 0, \quad S\eta\, U_{tz}(h, t) = P - M[\tau U_{ttt}(h, t) + U_{tt}(h, t)] \tag{8}$$

For $\rho Sh/M < 1$ and $4\tau S\eta/Mh > 1$ we can write the solution of Eq. (4), subject to the initial and boundary conditions (5)-(8), in the form

$$U(z, t) = \frac{Pz}{S\eta}(t + \tau) - \frac{Pz}{S\eta^2}\left[\frac{Mh}{S} + \frac{\rho h^2}{2} - \frac{\rho z^2}{6}\right] + \frac{P\rho h^3}{S\eta^2}\exp\left(-\frac{t}{2\tau}\right) \times$$
$$\times \sum_{k=1}^{\infty} \frac{\sin\beta_k \sin\beta_k z/h}{\beta_k^3(2\beta_k + \sin 2\beta_k)}\left[(3 - \xi_k^2)\cos q_k t + \frac{1 - 3\xi_k^2}{\xi_n}\sin q_k t\right] \tag{9}$$

where the β_k are the positive solutions of the equation

$$\operatorname{ctg}\beta = \frac{M}{S\rho h}\beta$$
$$\xi_k = \sqrt{\frac{4\tau\eta\beta_k^2}{\rho h^2} - 1}, \quad q_k = \frac{1}{2\tau}\xi_k \quad (k = 1, 2, 3, \ldots) \tag{10}$$

When $M \to 0$ the solution (9) agrees in form with the solution (23) of [7], obtained by an operational method; however, it differs from the latter in the coefficient of the exponential term. If in Eq. (9) we put $M = 0$ and let $\eta \to \infty$, then the solution (9) agrees with the solution of a similar problem [8] involving the motion of an absolutely elastic system.

For the given case the mass M is, in principle, never equal to zero; it can only be approximated in some manner to P/g. Therefore, it is of interest to consider the limit of the solution (9) for $M \neq 0$ as $\eta \to \infty$.

$$U(z, t) = \frac{Pz}{SG} - \frac{4Ph}{SG}\sum_{k=1}^{\infty}\frac{\sin\beta_k \sin(\beta_k z/h)\ \cos\beta_k (G/\rho h^2)^{-1/2} t}{\beta_k(2\beta_k + \sin 2\beta_k)} \tag{11}$$

The solution (9) applies in general for describing the motion of the disturbed mass of a Maxwell body under specified experimental conditions. It can be used to investigate materials of small viscosity and also when the influence of the moving system on the external shear force is negligible.

For stationary motion of the system the influence of the engineering instrumentation characteristics (of the mass M) is substantial for $t \gg 2\tau$

$$U(h, t) = \frac{Pht}{S\eta} - \frac{Ph}{S\eta^2}\left[\frac{Mh}{S} + \frac{\rho h^2}{3} - \frac{\eta^2}{G}\right] \tag{12}$$

It is evident that necessary readjustments must be made when dealing with actual characteristics (in engineering structures) of elastic and dissipative properties of the materials employed, particularly, when calculations for such structures are to be made.

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